esi® Engineering Bulletin

GENERAL

no. 41

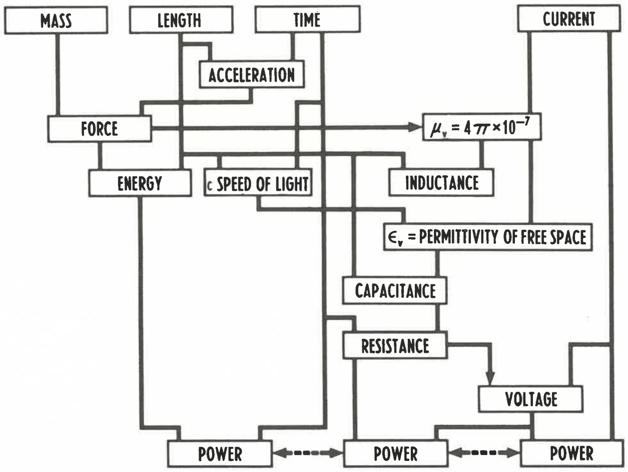
Electro Scientific Industries
13900 N.W. SCIENCE PARK DRIVE • PORTLAND, OREGON 97229

JULY 1964

DERIVATION OF ELECTRICAL UNITS FROM FUNDAMENTAL STANDARDS

by

R. D. KUYKENDALL and R. M. PAILTHORP



©Copyright 1964 Electro Scientific Industries, Inc.

LITHO IN U.S.A.

DERIVATION OF ELECTRICAL UNITS FROM FUNDAMENTAL STANDARDS

INTRODUCTION

Suppose that the electrical units were not planned, but just grew. (The supposition may not be far from the truth.) Suppose that each time a new unit was needed, it was selected in an arbitrary manner. The standard volt might be, for instance, the open-circuit voltage of a specified cell, the ampere might be the short-circuit current of a specific generator turned at a specified speed, and the ohm might be the resistance of a specified piece of wire at a specific temperature. One might easily think of other units for capacitance, inductance, etc.

CONSTANTS OF PROPORTIONALITY

The first thing one might notice would be that the electrical laws would all have constants of proportionality. Some examples of these laws are shown in Figure 1.

$$E = k_1 I R \qquad E_L = k_3 L \frac{dI_L}{dt}$$

$$P = k_2 I E$$

$$P = k_1 k_2 I^2 R \qquad I_C = k_4 C \frac{dE_C}{dt}$$

$$P = \frac{k_2}{k_1} \frac{E^2}{R} \qquad f_0 = \frac{1}{2\pi\sqrt{k_3 k_4 L C}}$$

FIGURE 1. ELECTRICAL LAWS WITH CONSTANTS OF PROPORTIONALITY

It is much more convenient to define the units in such a way that most of the constants are made equal to one. It is not possible to make them all equal one; for example, the constant 2π in the formula for resonant frequency is there because the formula has to do with sine functions, and would remain in the formula regardless of the values of k_3 and k_4.

CHOICE OF FUNDAMENTAL QUANTITIES

Note that there are four basically different formulae (hence four constants) and seven quantities: voltage, current, resistance, power, inductance, capacitance, and time (or frequency). If these were the only formulae necessary in electricity, we could establish three of these quantities as fundamental and define the rest of them in terms of the three fundamentals.

We could, for instance, choose arbitrary units of voltage, current and capacitance. Then, by setting all constants, k_n, equal to one, resistance, power and time are derived from the fundamentals and inductance could be derived from two fundamental units, voltage and current, and from one derived unit, time. If time is a derived quantity, we would have an awkward conversion factor when we tried to compare electrical time with the time that everyone else uses.

There are other quantities that would present awkward conversion factors if the fundamental units were not chosen with some care. Consider power and energy as an example. The chain of formulae that leads from the fundamental units of the mks (meter-kilogram-second) system to the definition of mechanical power is shown in Figure 2 and Table I.

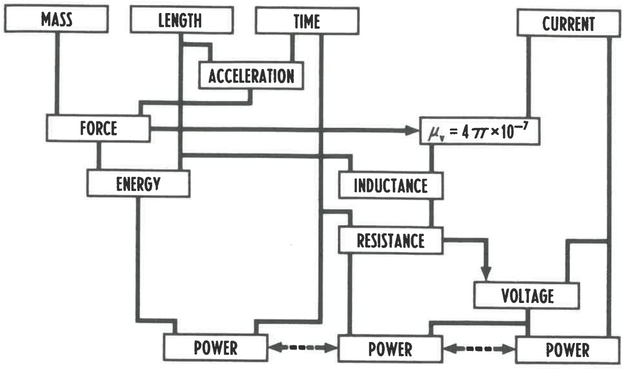


FIGURE 2. A DERIVATION OF POWER

QUANTITY	ABBREVIATION	FORMULA	NAME	SYMBOLIC UNITS
LENGTH	ℓ	(FUNDAMENTAL)	METER (m)	m
MASS	M	(FUNDAMENTAL)	KILOGRAM (kg)	kg
TIME	T	(FUNDAMENTAL)	SECOND (s)	s
ACCELERATION	a	$a = \frac{d^2\ell}{dT^2}$	—	$\frac{m}{s^2}$
FORCE	F	$F = Ma$	NEWTON (n)	$\frac{m\,kg}{s^2}$
ENERGY	w	$w = \int F d\ell$	JOULE (j)	$\frac{m^2 kg}{s^2}$
POWER	P	$P = \frac{dw}{dT}$	WATT (w)	$\frac{m^2 kg}{s^3}$

TABLE I. EXAMPLES OF QUANTITIES DERIVED FROM FUNDAMENTALS

CONSISTENT SYSTEM

Since power and energy can be derived both from the fundamental mechanical units of the mks system and from the electrical units, it would be well if both electrical and mechanical power and energy were measured in the same units. A system in which the same quantity always has the same units is called a consistent system. One might enforce this consistency by defining the watt as one of the fundamental units, but the present state of the art is such that power cannot be measured mechanically with satisfactory accuracy. For this reason the watt is not established as a fundamental. But we should stipulate that whatever system is used, units of power and energy must be consistent between mechanical and electrical systems.

MKSA SYSTEM

The difficulties of having an inconsistent system have been overcome in the MKSA (meter-kilogram-second-ampere) system of units, the system that has been adopted for most scientific work. This is accomplished by defining time to be fundamental in both electrical and mechanical measurements, and by defining the fundamental unit of current, the ampere, in a way which makes it possible to derive the electrical units from the fundamental mechanical units of length, time, and mass. In order to show how this is done, we introduce two more equations:

$$\textbf{Ampere's Equation:}\quad F = \frac{\mu_v I^2 \ell}{2\pi S}$$

gives the force, F, between two long straight conductors of length ℓ, placed a distance, S, apart in a vacuum when each conductor is carrying a current I. The symbol μ_v denotes the permeability of free space.

The other formula that we introduce:

$$L = \mu_v N^2 \pi r^2 \ell$$

gives the inductance, L, of a long solenoid of length ℓ, and radius r, with N turns per unit length, when it is in a vacuum.

Note that the permeability of free space is in both formulas and that the formula for inductance is in terms of permeability and length. If one quantity, permeability, inductance, or current is defined to be fundamental, the other two quantities can be derived from fundamentals, and all the other electrical quantities, such as voltage, capacitance, and resistance, can be derived from inductance, time, and current.

DEFINITION OF THE AMPERE

The International Conference on

Weights and Measures defined the Ampere in terms of Ampere's equation.

> "The ampere is the constant current which, if maintained between two straight parallel conductors of infinite length, of negligible circular sections, and placed 1 meter apart in a vacuum, will produce between these conductors a force equal to 2×10^{-7} mks units of force per meter of length."

The definition is in terms of force and length, but it implies an assumption that the permeability of free space, μ_v, is equal to $4\pi \times 10^{-7}$, and has caused a certain amount of confusion as to whether the ampere is fundamental or the permeability of free space is fundamental. There seems to be no overwhelming reason for calling one of them fundamental rather than the other as long as the two are properly interrelated.

There is at least one historical precedent for defining a unit in terms of a physical constant. When the metric system was first developed, the unit of mass, the gram, was defined to be the mass of a cubic centimeter of water. This definition simplifies the determination of density because it sets density equal to specific gravity. Implied in the definition is the assumption that the density of water is equal to one. One might say that the definition is actually a characteristic of water, its density. The definition of the fundamental unit of current, the ampere, is reminiscent of this definition of mass.

DEFINITION OF THE KILOGRAM

There is another parallel between the definitions of mass and current. An early definition of the ampere was: the current required to deposit a specified quantity of silver from an aqueous silver nitrate solution in one second. This definition of current was dropped because it was not possible to duplicate the experiment with sufficient accuracy. The original metric system definition of mass was dropped because of the difficulties of producing exactly a given volume of water. The kilogram is the fundamental unit of mass in the MKSA system, and is presently defined as the mass of a specific cylinder of metal.

DEFINITION OF THE METER

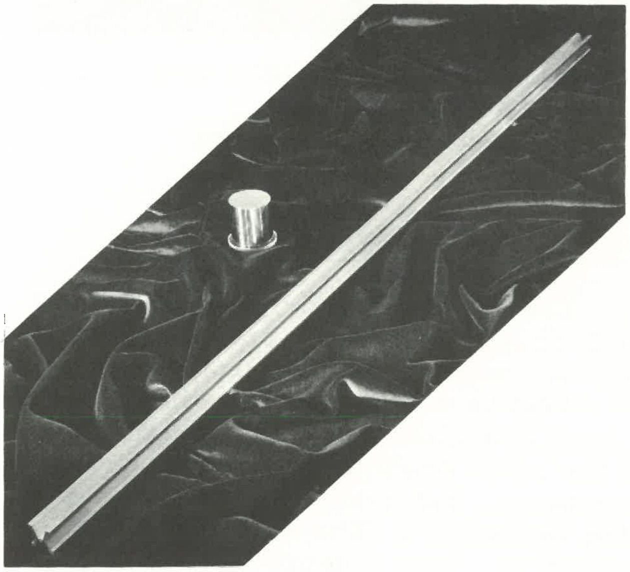


FIGURE 3. PROTOTYPE METER NO. 27 AND PROTOTYPE KILOGRAM NO. 20
(PHOTOGRAPH COURTESY NBS)

The other units of the MKSA system have had similar changes of definition since the beginning of the metric system. The meter was once defined as one ten-millionth of the distance from the equator of the earth to the pole. This distance could not be measured accurately enough, so the definition was changed to be the distance between two scratches on a specific platinum-iridium bar at 0°C. That definition was sufficient for more than a century and a half but recently the definition has been changed again because the phenomenon of interference fringes is more easily and accurately reproducible. The meter is now defined as 1,650,763.73 wavelengths of orange-red light emitted by ionized krypton-86 operating at the triple point of nitrogen.

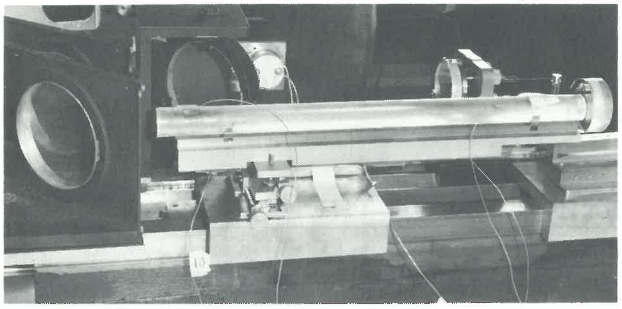


FIGURE 4. INTERFEROMETER USED WITH KRYPTON-86 LIGHT
(PHOTOGRAPH COURTESY NBS)

DEFINITION OF THE SECOND

The second was originally defined as 1/86,400 of a day. Unfortunately, time as measured by the rotation of the earth is not even as consistent as a good pendulum clock. The excursion of the earth around the sun is more consistent, but even this has measurable variations. The definition of the second has been amended several times in order to keep up with the state of the art of time measurement. At the present, the second is defined as: 1/31,556,925.9747 of the tropical year at 12^h ephemeris time, 0 January, 1900.

At the present, elapsed time is being measured, and the second is being maintained, by measurement of the resonant frequency at the ground state of the cesium atom. On occasion, this time is changed to agree with the results of astronomical observations. These observations relate the unit of time to its definition. It is quite likely that at some future date, the definition of the second will be in terms of the resonant frequency of some atom.

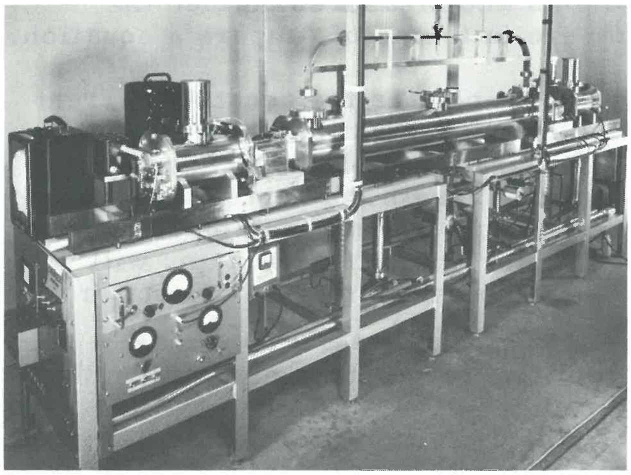


FIGURE 5. CESIUM BEAM FREQUENCY STANDARD
(PHOTOGRAPH COURTESY NBS)

No matter how convenient the definitions of the fundamental units are, it is of no use to define them if they cannot be reproduced. As we have seen, a convenient definition of the unit of mass based on a characteristic of water was discarded because of problems in accurate volume measurement. A convenient definition of the meter in terms of the earth's circumference was dropped because of problems of accurate measurement of the earth's circumference. The definition of the ampere in terms of electroplating a specified mass in a specified time was dropped because of problems in repeating the experiment.

DETERMINING THE AMPERE

At first glance, one might think that the definition of the ampere would be impossible to follow exactly. It specifies two straight parallel conductors of infinite length in a vacuum. It would seem that this would require an infinitely long straight laboratory in which to perform the experiment. The definition of the ampere was made for mathematical, not experimental, convenience. However, it is possible to construct apparatus for which the current can be calculated as a function of force. An apparatus used by the

National Bureau of Standards to produce a current that is known in terms of length and force is the NBS Current Balance.

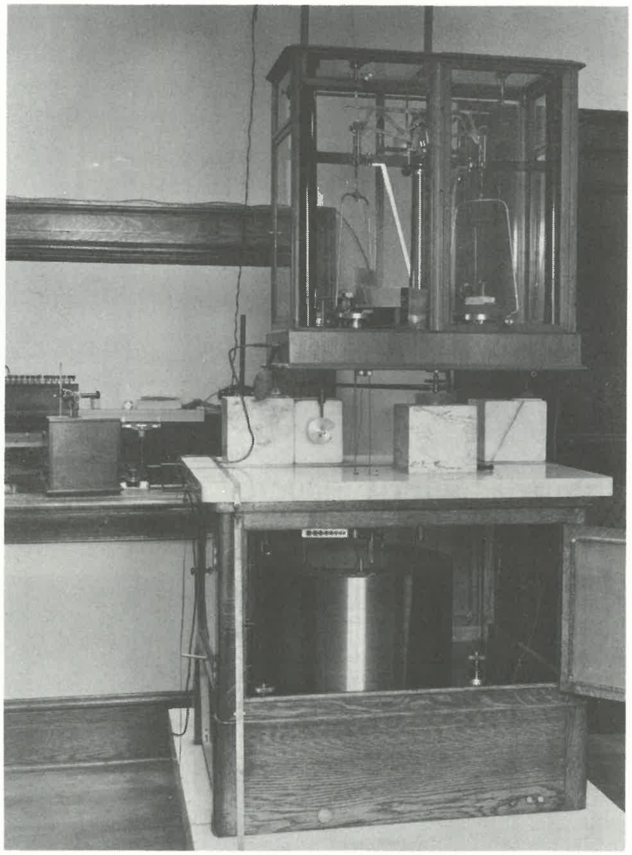


FIGURE 6. THE NBS CURRENT BALANCE (PHOTOGRAPH COURTESY NBS)

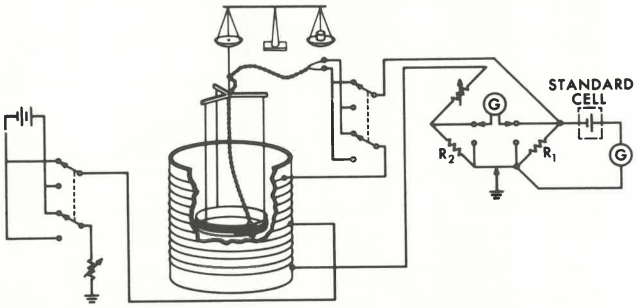


FIGURE 7. SCHEMATIC OF THE NBS CURRENT BALANCE

The current passing through the coils in the outer solenoid tends to move the center coil either up or down depending on the direction of current flow. The current is adjusted so that the force on the coil is balanced by weights in the pans of the scale, then the current in one coil is reversed and the scale is rebalanced by adding or removing a weight. The difference between the forces produced with the current flowing in each direction is proportional to the square of the current times a function of the mechanical dimension of the coils. The current passes through a standard resistor, R_I and the voltage drop is compared with that of a standard cell.

This experiment produces a known current, but when the experiment is over, the current is gone. The standard cell and the standard resistor are used in this experiment as a means of remembering what the current was.

The mathematics of computing the actual current are cumbersome and the procedure is time-consuming. For example, the function of the mechanical dimensions of coils requires exhaustive measurement of the length of the wires in each turn. The force produced by the coils is measured by balancing weights, and thus the force must be computed from the mass of the weights and the acceleration of gravity. It is necessary for the accuracy of the experiment to determine the exact acceleration of gravity, which, by the way, varies with the tides. The Bureau of Standards' estimation of the accuracy of the derivation of current is shown in Table II.

DERIVATION OF THE VOLT

It might be noted at this point that the standard cell and standard resistor used in the current derivation experiment do not need to have any particular accuracy. The experiment determines their ratio ($I = \frac{E}{R}$) and gives no indication of their values or accuracy. Once

SOURCES OF ERROR	e (50% ERROR)
MEASUREMENT OF THE FORCE	2 ppm
LEAD CORRECTIONS	2
RADIUS OF FIXED COIL	2
RADIUS OF MOVABLE COIL	3
CURRENT DISTRIBUTION OVER WIRES	1
CALIBRATION OF LENGTH STANDARDS	1
CALIBRATION OF STANDARD MASS	1
CALIBRATION OF ELECTRICAL STANDARDS	1
ADJUSTMENT OF COILS	1
TEMPERATURE OF COILS	1
PERMEABILITY OF FORMS	<1
ACCELERATION OF GRAVITY	3
50% ERROR IN FINAL RESULT, $\sqrt{\Sigma e^2}$	6

TABLE II. SOURCES OF ERROR CAUSING AN UNCERTAINTY, e, IN RATIO I_{abc}/I_{NBS} AS LARGE AS 1 ppm

this ratio is known, derivation of the ohm will also yield the value of the volt.

THE OHM AND INDUCTANCE

The original definition of the ohm was the resistance of a column of mercury of a specific length and diameter. This definition was not sufficiently reproducible and is not conducive to a consistent system. However, the definition of current as given implies a method of deriving the ohm. If the permeability of free space is known, we have seen that it is possible to compute the inductance of a coil from its dimensions. If inductance is known, it is theoretically possible to derive resistance through inductance.

INDUCTANCE

Figures 8 and 9 show the mutual and self inductors that the National Bureau of Standards has constructed. The physical dimensions of the inductors have been measured and the inductances calculated.

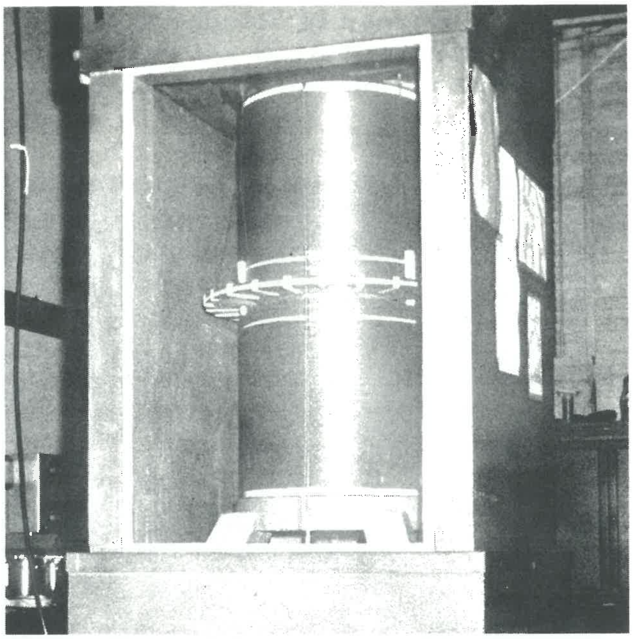


FIGURE 8. MUTUAL INDUCTOR

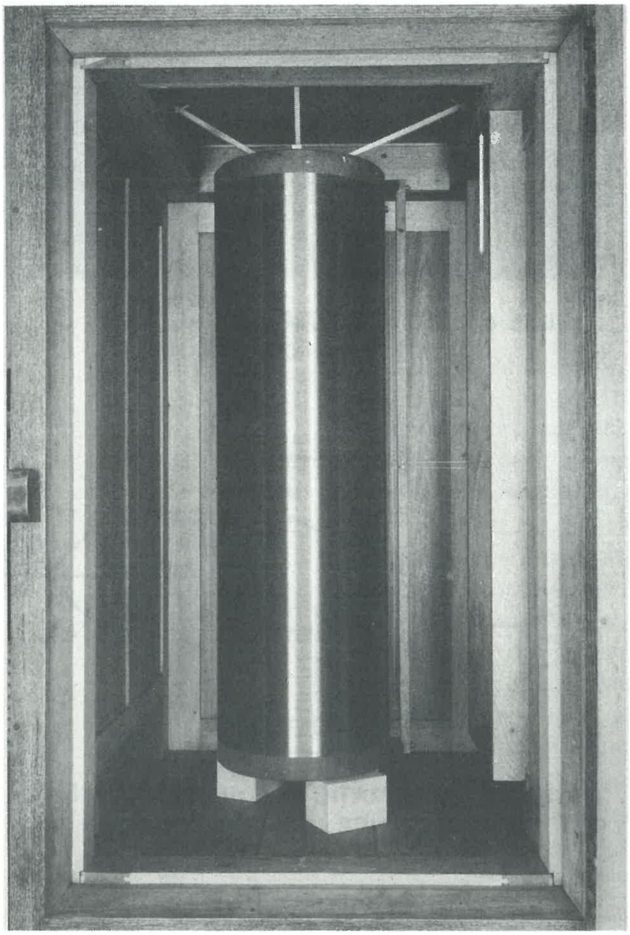


FIGURE 9. SELF INDUCTOR
(PHOTOGRAPH COURTESY NBS)

DERIVING THE OHM

The mutual inductor and the circuit shown in simplified form in Figure 10 provide a method of determining the resistance of a resistor in terms of the dimensions of the mutual inductor and the frequency of the commutator.

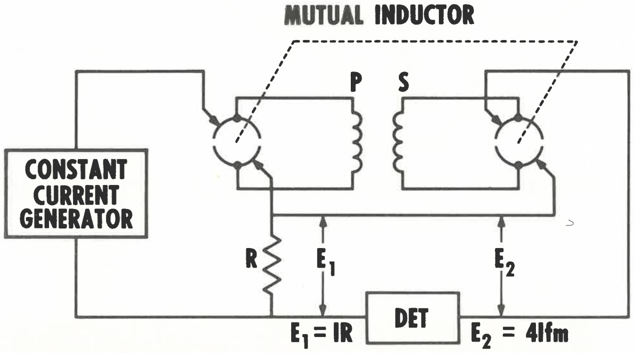


FIGURE 10. SIMPLIFIED SCHEMATIC OF WENNER METHOD OF DETERMINING RESISTANCE

The primary commutator reverses the voltage across the primary of the mutual inductor. The voltage reversals cause an alternating current in the secondary, and the current is rectified by the secondary commutator which rotates synchronously with the primary commutator. If the current fed to the primary commutator is constant, the average voltage across the secondary commutator, E_2, is:

$$E_2 = 4Ifm$$

where I is the primary current, f is the commutator frequency, and m is the mutual inductance.

The resistance or the frequency are adjusted to make the galvanometer read zero, in which case the voltage across the resistor, E_1, is equal to E_2.

Since:

$$E_1 = IR$$

and

$$E_2 = 4Ifm$$

at balance

$$E_1 = E_2$$

so that

$$IR = 4Ifm$$

or, dividing by I;

$$R = 4fm$$

The measurement of frequency in terms of atomic resonance can be made several orders of magnitude more accurately than the other factors of the equation, and the mutual inductance is determined by mechanical measurements. From these, the resistance can be calculated.

Of course, the actual circuit is not so simple. Actual circuits never seem to have the classic simplicity that the fundamental circuits do.

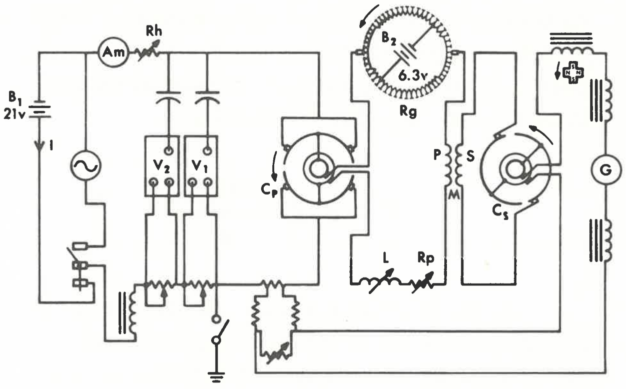


FIGURE 11. COMPLETE SCHEMATIC OF WENNER METHOD

The major complications involve the regulation of the primary current, the production of wave forms that have predictable shape, and delivery of sufficiently smooth average voltage to the galvanometer. With this method, the National Bureau of Standards has determined the absolute value of the ohm with an estimated accuracy of 5 parts per million.

THE OHM AND CAPACITANCE

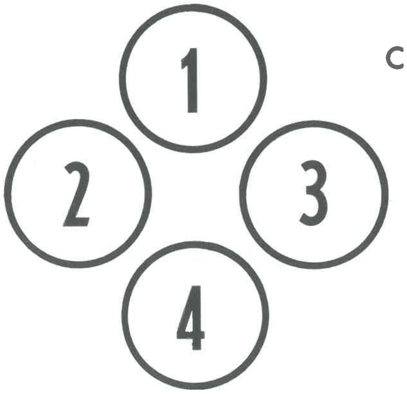
Another way to find the absolute value of the ohm is to build a capacitor with a capacitance that can be calculated from its physical dimensions and build a circuit that compares resistance with capacitive reactance at a known frequency. This is the method currently used by the National Bureau of Standards.

CAPACITANCE

A. M. Thompson and D. G. Lampard of the Australian National Standards Laboratory have developed a type of capacitor for which the capacitance can be calculated more accurately than any previous design. A theorum published by Lampard in 1956 shows that a cross-capacitor constructed with electrodes that are parallel cylinders of arbitrary cross section, and so constructed that the cross-capacitances are equal, will have cross-capacitances that can be calculated from the single measurement of length, **S**, by the formula:

$$C = \epsilon_v \ell \frac{\log_e 2}{\pi}$$

A particular case of this theory is illustrated by a cross-capacitor constructed of circular cylinders, as shown in Figure 12. If Electrodes **2** and **3** are grounded, the capacitance between Electrodes **1** and **4** is measured, and then Electrodes **1** and **4** are grounded and the capacitance between Electrodes **2** and **3** is measured. As the two capacitances measured in this way approach equality, the value of either capacitance approaches a constant value per unit length. This constant value is approximately 2 picofarads per meter.



$$C = \epsilon_v \ell \frac{\log_e 2}{\pi}$$

FIGURE 12. THOMPSON-LAMPARD COMPUTABLE CAPACITOR

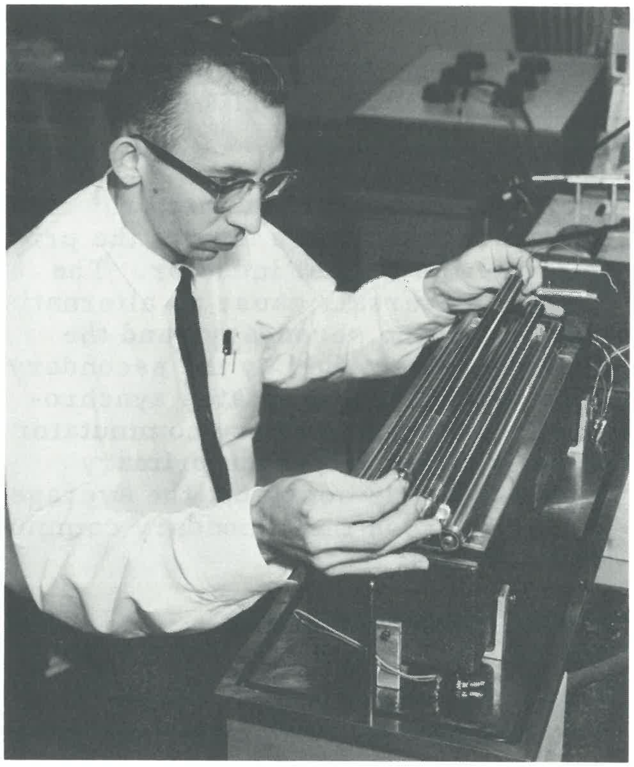


FIGURE 13. THE NBS COMPUTABLE CAPACITOR
(PHOTOGRAPH COURTESY NBS)

It should be noted here that one of the factors of the constant value per unit length of a calculable capacitor is the permittivity of free space, ϵ_v. Permittivity, ϵ_v, and permeability, μ_v, of free space are related as follows:

$$\epsilon_v = \frac{1}{\mu_v C^2}$$

where c is the speed of light. As stated before, permeability of free space is defined to be equal to $4\pi \times 10^{-7}$. If the speed of light were known exactly, it would be possible to find the permittivity of free space exactly. This means that the accuracy of the derivation of capacitance depends on the accuracy with which the speed of light is known.

ANOTHER WAY TO DERIVE THE OHM

With a circuit similar to that shown in Figure 14, one can compare capacitive reactance and ac resistance. In order to do so, the two generator voltages must be exactly 90° out of phase.

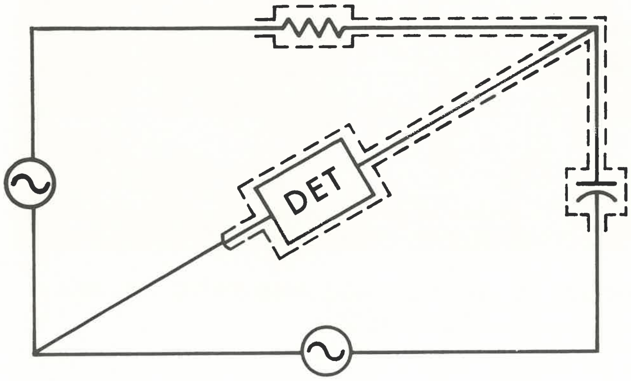


FIGURE 14. CAPACITIVE REACTANCE BRIDGE SCHEMATIC

In order to correct for several effects, including errors of phase and residual capacitances and resistances, the National Bureau of Standards has performed the experiment of comparing capacitive reactance and ac resistance using the frequency-dependent bridge shown in Figures 15 and 16.

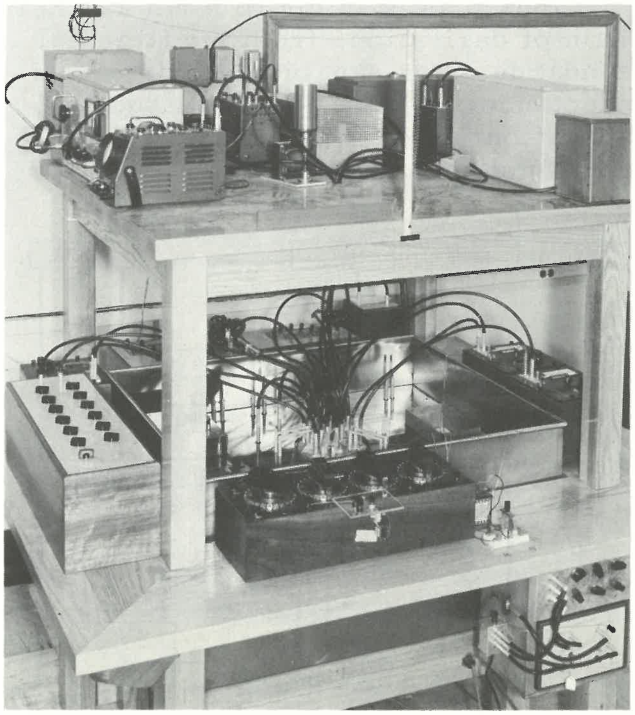


FIGURE 15. THE NBS FREQUENCY-DEPENDENT BRIDGE
(PHOTOGRAPH COURTESY NBS)

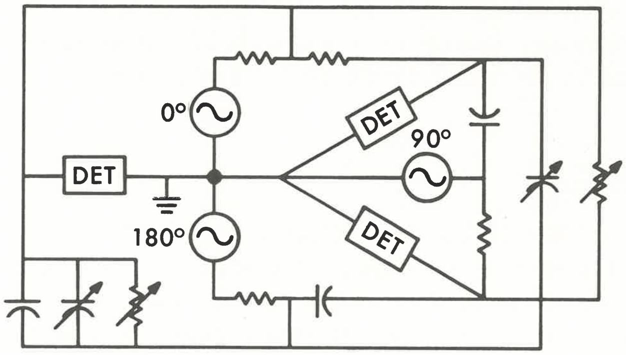


FIGURE 16. FREQENCY-DEPENDENT BRIDGE SCHEMATIC

AC TO DC TRANSFER

Once the ac resistance has been derived, it is necessary to transfer to dc resistance. This is accomplished by using a special resistor which has very low capacitive and inductive residuals. The National Bureau of Standards estimates that the probable accuracy of this method of deriving dc resistance through capacitance is ±2.1 parts per million, excluding errors caused by uncertainty of the exact speed of light. The speed of light is believed to be known within one part per million.

Thus it is possible to form another chain of derivations from fundamental standards using capacitance instead of inductance. Figure 17 shows such a derivation; another derivation of units from fundamental standards.

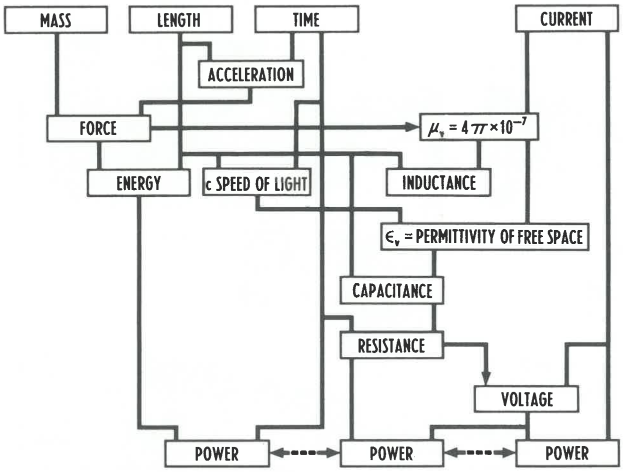


FIGURE 17. A DERIVATION OF UNITS USING CAPACITANCE

Other electrical quantities, such as voltage, must be transferred from ac to dc or vice versa. For this reason, there is a branch of measurement devoted to ac to dc transfer.

RATIOS

We have seen that the MKSA values for voltage, resistance, current inductance, and capacitance can be derived from the fundamental standards of length, mass, and time, and as a matter of fact, inductance and capacitance can be computed solely from dimensional measurements. These computations are based on the assumption that is implied in the definition of current: that the permeability of free space is $4\pi \times 10^{-7}$.

In general the devices and experiments used to relate electrical and mechanical measurements give one result per experiment. The result is a current of a specific size, a certain inductance, a specific capacitance, or a resistance of a certain value. The next problem is to find multiples of each known value. This problem is similar to the problem of manufacturing a set of weights if you are given only a kilogram as a start.

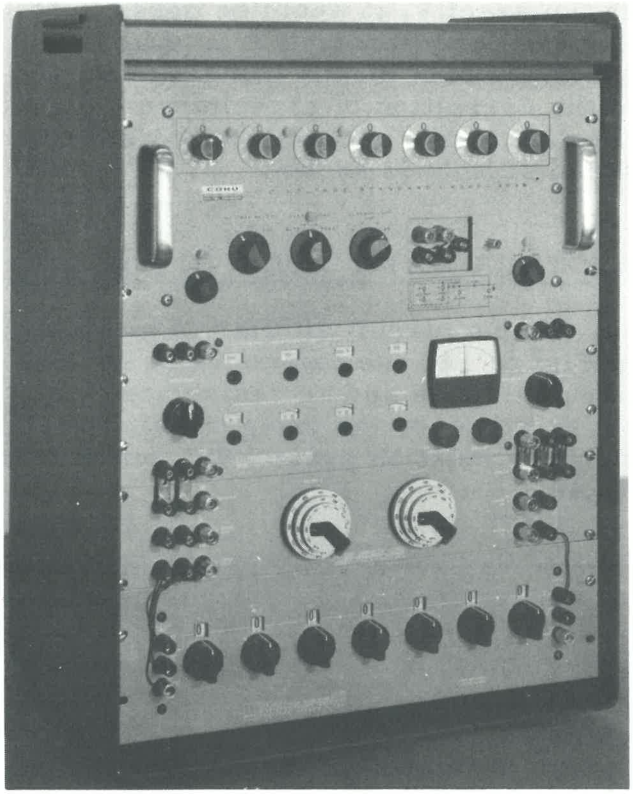


FIGURE 18. DC VOLTAGE AND RATIO CALIBRATION SYSTEM

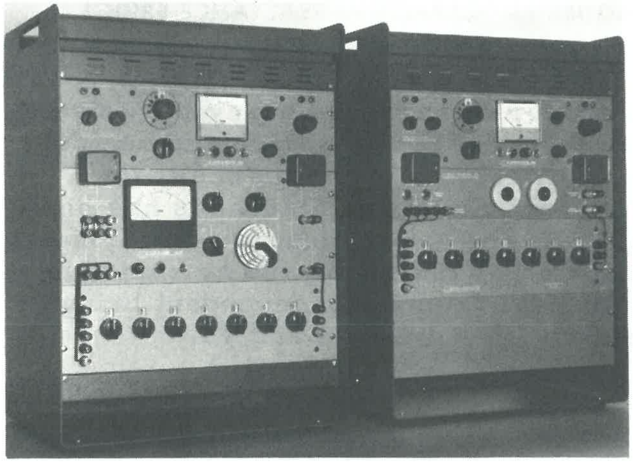


FIGURE 19. AC RATIO CALIBRATION SYSTEMS

Many electrical quantities, such as voltage, resistance, current, capacitance, and inductance, can be transferred to different values by using known resistance ratios or transformer ratios. Since the ratios do not involve the accuracy or value of the electrical unit, the ratios can be established in any laboratory which has the proper equipment and techniques. Some ratio equipment is shown in Figures 18 and 19.

Since the accuracy of the methods of comparing different values of electrical quantities is quite high, the trend these days is for companies in the electronics industry to send a few transportable reference standards of high stability to the National Bureau of Standards for calibration and then compare them to working standards using ratio techniques. Some examples of transportable reference standards are shown in Figure 20.

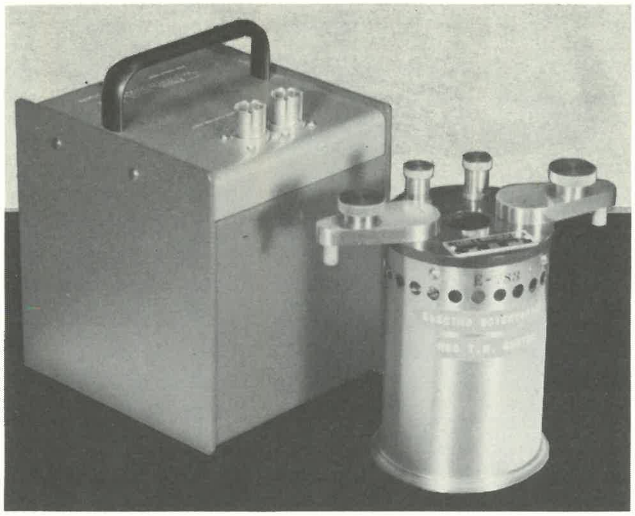


FIGURE 20. TRANSPORTABLE REFERENCE STANDARDS

LEGAL VALUE OF UNITS

One might note that the methods of comparing resistors to resistors, voltages to voltages, capacitors to capacitors, etc. are more accurate than the absolute values of the quantities measured are known. For instance, one can compare the values of two resistors with an accuracy of 0.1 part per million, but the absolute value of the ohm, the value derived from the fundamental quantities of length, mass, time and current is only known within a few parts per million. It is desirable to have national reference standards that are as accurate as their values can be intercompared. Therefore, the units derived and maintained by the National Bureau of Standards are defined by act of Congress to be perfect for purposes of trade and are the legal values of the units. As the NBS develops more accurate methods of deriving the absolute values of the units, they will make appropriate changes, if required, in the legal values of the national reference standards.

REFERENCES

Driscoll, R. L. and Cutkosky, R. D., Measurement of Current with the National Bureau of Standards Current Balance, Journal of Research of the National Bureau of Standards, Vol. 60, No. 4, April 1958.

McGregor, M. C., Hersh, J. F., Cutkosky, R. D., Harris, F. K. and Kotter, F. R., New Apparatus at the National Bureau of Standards for Absolute Capacitance Measurement, IRE Transactions on Instrumentation, Vol. I-7, Numbers 3 and 4, Dec. 1958.

McNish, A. G., The Basis of our Measuring System, Proceedings of the IRE, Vol. 47, No. 5, Part 1, May 1959.

McNish, A. G., Fundamentals of Measurement, Electro-Technology, Vol. 71, No. 5, May 1963.

Silsbee, F. B., The Ampere, Proceedings of the IRE, Vol. 47, No. 5, Part 1, May 1959.

Silsbee, F. B., Establishment and Maintenance of the Electrical Units, NBS Circular 475.

Silsbee, F. B., Extension and Dissemination of the Electrical and Magnetic Units by the National Bureau of Standards, NBS Circular 531.

Varner, Wm. R., The Fourteen Systems of Units, O.S.C. Cooperative Association, Corvallis, Oregon.